just the identity transformation; the generalized covariant Dirac position operator is just the canonical coordinate $\mathbf{x}=\mathbf{q}$. So the generalized inverse Foldy-Wouthuysen transformation leaves the Hamiltonian in the canonical form (4.1) for zero spin. But considerably more manipulation is needed to get the Klein-Gordon equation.[8]

The equations (5.6) can be put directly into the Schrödinger equation form (5.4) with a six-component wave function whose components are $\mathbf{A}$ and $m^{-1}\mathbf{E}$.[8,11,21] The Hamiltonian for this equation is not Hermitian but is pseudo-Hermitian in the appropriate indefinite metric.[13] From our point of view, the Schrödinger equation with the Hamiltonian (5.3), which is obtained by the generalized inverse Foldy-Wouthuysen transformation from the canonical Hamiltonian (4.1), is more nearly the spin 1 analog of the Dirac equation. Whether it will be more useful remains to be seen.

Various authors have developed the transformation which connects the canonical Hamiltonian (4.1) directly to the Schrödinger equation form of Eqs. (5.6) with the wave function whose components are $\mathbf{A}$ and $m^{-1}\mathbf{E}$.[8,13] This is not a unitary transformation but is pseudo-unitary in the appropriate indefinite metric.[13] From our point of view, it appears as the combination of the generalized inverse Foldy-Wouthuysen transformation and the manipulations (5.5). This transformation does not have all of the properties of the Foldy-Wouthuysen transformation and cannot be put to all of the same uses. For example, if we use it to transform the position operators $\mathbf{x}$ which appear as the independent variables in the equations (5.6) to the representation in which the Hamiltonian has the canonical form (4.1), we get an operator[14] which does not satisfy the last of Eqs. (B), the condition for Lorentz covariance.

Finally, we want to point out that we have exposed several simple features of the spin 1/2 situation which are not shared by spin 0 or 1. In particular, the inverse Foldy-Wouthuysen transformation for $s=1/2$ gives us the local invariant Dirac equation. The analogous equations for $s=0$ and 1 are not local.

ACKNOWLEDGMENT

We are grateful to Professor E. C. G. Sudarshan for helpful discussions of this work.

[12] E. M. Corson *Tensors, Spinors, and Relativistic Wave Equations* (Hafner Publishing Company, New York, 1953), especially Secs. 26(b) and 39(d)(i).

[13] K. M. Case, Phys. Rev. **95**, 1323 (1954).

[14] K. M. Case, Ref. 13, Eq. (38).

PHYSICAL REVIEW VOLUME 132, NUMBER 4 15 NOVEMBER 1963

Improvement of the Born Series at Low Energy*

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(Received 8 July 1963)

The Born series for a quantum-mechanical Green function is studied. A prescription is given for making "best" use of the information contained in the first few terms of that series, and, in particular, for calculating bound states or resonances from them. This prescription is based on heuristic convergence arguments whose formal steps are somewhat reminiscent of renormalization group methods. The present considerations may be applied to potential scattering as well as to quantum field theory. They are expected to be valid for low-energy phenomena and finite-range forces. The prescription is tested, using only the first two Born terms, in the case of a nonrelativistic particle moving in a Yukawa potential: For well depths producing a single shallow bound state, the usual effective-range results are closely reproduced, and, in some ways, improved upon.

1. INTRODUCTION

THE problem of replacing the quantum-mechanical Born (or perturbation) series by a more convergent expansion is already a well-studied one. Its importance arises from the fact that many cases of great physical interest cannot be treated by means of that series. Bound-state and resonace problems fall in this category whenever the unperturbed problem yields only a continuum of noninteracting states. This is precisely the situation one must face in relativistic field theory. The current experimental results with strongly interacting particles only emphasize the need for improved calculational procedures in this area.

Among the many existing approaches[1–3] to the ques-

* Research supported by the National Science Foundation.

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[1] See, for example, P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Part II, Chap. 9.

[2] For recent approaches see M. Rotenberg, Ann. Phys. (N. Y.) **21**, 579 (1963), and its bibliography. Rotenberg's method, like the present one, is based on a regrouping of the Born terms, designed to accelerate convergence. His "regrouped" Eq. (59), in particular, should be compared with our formulation, in which a true regrouping only occurs as an intermediate step.

[3] A method of circumventing the convergence difficulty of

tion, some, like the Fredholm or Feenberg methods,[1] while successful in potential problems, have not so far been readily applicable to relativistic field theory. In the latter context we should only like to mention the renormalization group method[4] as possibly having some relation to the ideas suggested here. As far as could be ascertained, however, neither that method nor (in general) its results coincide with ours. Nevertheless, a common starting point is the recognition that a renormalization constant (or a coupling constant) may be specified in a continuously infinite number of ways by stating the numerical value of some Green function at a chosen point in momentum space (the "renormalization momentum").

The present article, which is only intended as an exposition of the method, will be confined to potential scattering for definiteness of presentation. It will be readily apparent, nevertheless, that the arguments are of wider generality.

2. THE BORN SERIES

Since the method to be presented makes use of the standard Born series as a starting point, we briefly review this expansion to establish a notation.

Consider an integral equation of the type

$$G(s)=G_0(s)+g(KG)(s)\,, \tag{2.1}$$

where $G(s)$ is a single unknown function of a finite set, s, of continuous variables s_1, s_2, $\cdots$, and where K is a given linear operator which maps G into some other function KG. The function $G_0(s)$ is also given; the real parameter g is adjustable. For the purpose of avoiding undue generality one may think of gK as being the integral operator associated with a certain Schrödinger equation and incorporating certain boundary conditions; G is Green's function for the problem. The variables s might be positions, momenta, etc., according to the representation chosen. For example, let

$$i\dot{\psi}+(2m)^{-1}\nabla^2\psi-gV\psi=0 \tag{2.2}$$

be the Schrödinger equation considered. Let

$$V(\mathbf{r})=(2\pi)^{-3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{r}}W(\mathbf{p})\,. \tag{2.3}$$

Then, the retarded Green function, suitably defined in energy-momentum space, satisfies the integral equation

$$G(\mathbf{q},\mathbf{p},E)=-(2\pi)^3\delta(\mathbf{p}-\mathbf{q})+g(2\pi)^{-3}\times\int d^3k[E-(2m)^{-1}k^2+i\epsilon]^{-1}\times W(\mathbf{q}-\mathbf{k})G(\mathbf{k},\mathbf{p},E)\,. \tag{2.4}$$

This provides a detailed instance of (2.1).

The Born series for G consists of the Taylor expansion

$$G=\sum_{n=0}^{\infty}g^nG_n\,. \tag{2.5}$$

Whether or not this series converges, it may be used to define the sequence of functions G_n. We have, in fact,

$$G_n=K^nG_0\,. \tag{2.6}$$

3. MODIFIED BORN SERIES

In order to find an expansion for G with optimized convergence properties we now proceed in two steps. First, we merely replace one Taylor expansion by another. For this purpose we select a new expansion parameter λ which will be considered a function of g, and vice versa:

$$g=g(\lambda)\,. \tag{3.1}$$

Now

$$G=\sum_{n=0}^{\infty}\lambda^nF_n\,. \tag{3.2}$$

This form is obtained if, as we assume, g can itself be expanded as a Taylor series in λ. We somewhat specialize this series to be

$$g=\lambda(1+\sum_{n=1}^{\infty}g_n\lambda^n)\,, \tag{3.3}$$

where the g_n are numerical coefficients to be chosen presently.

It will be advantageous to rewrite (3.2) as

$$G=G_0+G_1\sum_{n=1}^{\infty}\lambda^nA_n\,, \tag{3.4}$$

where the functions A_n are now independent of any arbitrary multiplicative number (or function) entering into the definition of G. We consider regions of the variables in which G_1 does not vanish. Inserting (3.3) into (2.5), we find

$$\begin{aligned}&A_1=1\,,\\&A_2=G_2/G_1+g_1\,,\\&A_3=G_3/G_1+2g_1G_2/G_1+g_2\,,\quad\text{etc.}\end{aligned} \tag{3.5}$$

We can now enforce the *rapid* convergence of the new series (3.4) for *all* values of λ at some chosen point $s=\sigma$.

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bound states and resonances has been suggested by S. Weinberg, Phys. Rev. **130**, 776; **131**, 440 (1963). He attempts to replace the problem by a physically equivalent one where each resonance and bound state is already made to occur in the unperturbed problem as a "particle" in its own right.

[4] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), Chap. VIII.

We only have to choose $g_1, g_2, \cdots$ such that

$$A_2(\sigma)=A_3(\sigma)=\cdots=0. \tag{3.6}$$

Explicitly,

$$\begin{aligned} g_1&=-G_2(\sigma)/G_1(\sigma), \\ g_2&=-G_3(\sigma)/G_1(\sigma)+2[G_2(\sigma)/G_1(\sigma)]^2, \quad \text{etc.} \end{aligned} \tag{3.7}$$

At this point it should be observed that nothing is known or assumed about the convergence of the series (3.3) for g. It only serves the formal purpose of obtaining the functions $g_n(\sigma)$.

Although (3.4) now converges at σ about as well as a series can (it is reduced to its first two terms), we have no guarantee that it converges at any $s\neq\sigma$. We make, however, the working hypothesis that it does converge in a finite, but possibly small region S of s which includes σ. (S may, of course, shrink as $|\lambda|$ increases.) If s is in S, we now expect the function G to be adequately represented by (3.4), with the A_n being given by (3.5). The dependence of the series on s and σ is worth exhibiting explicitly; for convenience we define the function

$$\tilde{G}(s)=[G(s)-G_0(s)]/G_1(s). \tag{3.8}$$

Then

$$\begin{aligned} \tilde{G}(s)=\lambda(\sigma)&+\lambda^2(\sigma)[G_2(s)/G_1(s)+g_1(\sigma)] \\ &+\lambda^3(\sigma)[G_3(s)/G_1(s)+2g_1(\sigma) \\ &\times G_2(s)/G_1(s)+g_2(\sigma)]+\cdots. \end{aligned} \tag{3.9}$$

If this rearrangement is to accomplish anything, not all values of σ are equally suitable. This may be seen as follows: Since the constant g is real, its variation truly represents a single degree of freedom. Consequently, there is no loss of generality in assuming that λ is real in (3.1). Equation (3.3) then shows that the g_n must be real. We then see from (3.7) that all the $G_n(\sigma)$ $(n\geqslant 1)$ must have the same complex phase. This only happens in certain regions (to be called constant-phase regions) of the variable σ, if at all. For example, in energy-momentum space we must not consider values of σ corresponding to a physical scattering process, although subthreshold real energies are usually all right.[5] Similarly, if partial-wave amplitudes are considered, we must take σ to correspond to a real energy lying between the left- and right-hand cuts.[6] These remarks will be illustrated at a later stage.

We finally observe that, in (3.9), the parameter λ must itself be considered a function of σ. Indeed, g is clearly independent of σ. Since the g_n are functions of σ it follows from (3.3) that such is also the case for λ.

4. A SERIES INDEPENDENT OF THE COUPLING STRENGTH

The second step in the procedure is to abolish, using continuity arguments, the privileged nature of the point $s=\sigma$. Let us consider from now on the dependence of any function upon, say, s_1 (or σ_1) only, keeping $s_2, s_3, \cdots$ ($\sigma_2, \sigma_3, \cdots$) as fixed parameters. Dropping the subscript 1 for convenience, we shall now speak of the single variable s (or σ).

We assume σ to have been chosen at a point where the function $\tilde{G}$, defined by (3.8), is differentiable. Then, inside S, the derivative $\tilde{G}'(s)$ is also a convergent series in λ. In particular, we can compute

$$\tilde{G}'(\sigma)\equiv[\tilde{G}'(s)]_{s=\sigma} \tag{4.1}$$

as a convergent series from (3.9). Finally, we can eliminate λ by the exact relation

$$\lambda=\tilde{G}(\sigma). \tag{4.2}$$

The result is

$$\begin{aligned} \tilde{G}'=\tilde{G}^2(G_2/G_1)'&+\tilde{G}^3[(G_3/G_1)'-2(G_2/G_1)(G_2/G_1)'] \\ &+\tilde{G}^4\{(G_4/G_1)'-3(G_2/G_1)(G_3/G_1)' \\ &+[5(G_2/G_1)^2-2G_3/G_1](G_2/G_1)'\}+\cdots, \end{aligned} \tag{4.3}$$

where all quantities are evaluated at the same point σ (or s), and where we have included one more explicit term than in (3.9). By requiring (4.3) to hold everywhere, we eliminate the need for singling out a special point. Thus, a first-order nonlinear differential equation has been obtained for the unknown function $\tilde{G}(s)$ in terms of the known functions $G_1(s), G_2(s), \cdots$ and their first derivatives. This equation no longer contains any coupling parameter. Therefore, the latter can only be hidden in the constant of integration for $\tilde{G}$. The presumed advantage of (4.3) over the Born series is that the right side has now been expressly designed for optimum convergence in constant-phase regions. In or near such regions, (4.3) is, in fact, proposed as a substitute for the Born series. In practice, Green functions have such an analytic structure,[5,6] that the procedure should lend itself to the calculation of low-energy scattering and shallow bound states.

5. SECOND-ORDER APPROXIMATION

If all the terms of (4.3) are neglected, i.e., if the right side is approximated by zero, we obtain for G the first Born approximation, the constant of integration playing the role of the coupling parameter.

The simplest nontrivial application of our procedure involves the first term of (4.3). To this order,

$$\tilde{G}'=\tilde{G}^2(G_2/G_1)', \tag{5.1}$$

which may be integrated and solved for G to yield

$$G=G_0+[\kappa-G_2/G_1]^{-1}G_1, \tag{5.2}$$

where κ is the constant of integration; κ^{-1} plays the role of an effective coupling constant. Comparison with the Born series shows that $\kappa^{-1}\to g$ as $g\to 0$. To see in what sense κ is a constant, suppose the derivatives in (5.1) are taken with respect to s_1. Then κ must be independent of s_1. If the method is to be exploited to best advantage, the argument should be carried out simultaneously for

[5] N. N. Khuri, Phys. Rev. **107**, 1148 (1957).
[6] R. G. Newton, J. Math. Phys. **1**, 319 (1960).

all variables $s_1, s_2, \cdots$. Consequently, κ should be a true constant, in the sense that it should not depend upon any dynamical variable.

The remarkable fact that (5.1) could be integrated without detailed knowledge of G_1 and G_2 does not carry over to higher orders of approximation. Also, to higher orders it may no longer be possible to treat all variables $s_1, s_2, \cdots$ on the same footing. A selection will then have to be made of whatever variable is of greatest physical interest in the problem considered. In this article, we do not further discuss the higher-order approximations, but confine the subsequent remarks to the second-order form (5.2). This form depends on the representation chosen for G. Consider, for definiteness, the scattering amplitude obtained from G by specializing to the energy shell:

$$p^2=q^2=2mE, \tag{5.3}$$

and assuming a spherically symmetric potential:

$$G\rightarrow M(\Delta^2,E) \tag{5.4}$$

(momentum representation),

$$G\rightarrow N(l,E) \tag{5.5}$$

(partial-wave representation). Here

$$\Delta^2=(\mathbf{p}-\mathbf{q})^2 \tag{5.6}$$

is the momentum-transfer squared, while l is the angular-momentum quantum number. The unscattered amplitude G_0 in (5.2) is ignored. In the momentum representation, (5.2) becomes

$$M(\Delta^2,E)=[\kappa_M-M_2/M_1]^{-1}M_1, \tag{5.7}$$

where M_1, M_2 are the first two Born terms of M, while similarly, in the partial-wave representation,

$$N(l,E)=[\kappa_N-N_2/N_1]^{-1}N_1. \tag{5.8}$$

The constants of integration κ_M, κ_N are not *a priori* the same. As recent analyticity studies have emphasized,[7] the variable l may be treated as continuous. This motivates us to make κ_N independent of l, as well as of E. It is clear that, if M and N are taken from the approximations above, then (setting $\mathbf{p}\cdot\mathbf{q}=p^2\cos\theta$)

$$M\neq\sum_{l=0}^{\infty}P_l(\cos\theta)N(l,E), \tag{5.9}$$

in general, even though

$$M_n=\sum_{l=0}^{\infty}P_l(\cos\theta)N_n(l,E), \tag{5.10}$$

by definition. The inequality (5.9) illustrates what is meant by the representation dependence of the second-order approximation. Thus, some judgment must be exercised as to the representation which is adequate to a given problem. It is likely, for example, that (5.8) is more suited to the calculation of bound states, while (5.7) is indicated in the study of total cross sections.

Although this article will not go into detailed field-theoretical applications, the following remarks can be made. In the Lee model,[8] (5.2) is exact for N-θ scattering. (There is no distinction between momentum and partial-wave representations here.) A similar result applies in simple cases of separable potentials. In more realistic models, it should still be noted that the coupling renormalization is embodied in (5.2). Consider the prevalent situation[9] where

$$G_2=G_2^{\text{reg}}+CG_1, \tag{5.11}$$

G_2^{reg} being cut-off independent, while the constant C diverges for infinite cutoff. It is then sufficient to make the difference κ-C convergent; (5.2) is applicable without modification.

Finally, consider a case where the analysis of Sec. 3 breaks down: suppose $G_1\equiv 0$. This may happen, for example, in the $\lambda\phi^4$ model (with isospin) if we consider the two-particle system with $T=1$. Then the first Born term contributes nothing. By a simple adaptation of the preceding arguments one can easily derive, to second nonvanishing order, the formula

$$G=G_0+[\kappa-\tfrac{1}{2}G_3/G_2]^{-2}G_2. \tag{5.12}$$

6. APPLICATION TO THE YUKAWA POTENTIAL

Since the foregoing considerations are heuristic rather than mathematically rigorous, it is essential to test them in practice. For this purpose we select the Yukawa potential model, whose Green function has manageable Born terms. This model bears a reasonably close relation, both physically and mathematically, to field-theoretical interactions. Furthermore, nearly exact solutions are already available in the literature for comparison.

In the notation of Sec. 2, let

$$V(\mathbf{r})=(4\pi r)^{-1}e^{-\mu r}, \tag{6.1}$$

$$W(\mathbf{p})=(p^2+\mu^2)^{-1}. \tag{6.2}$$

The second-order formulas, (5.7) and (5.8), are the ones to be tested in this section. To this order, our ambition should not go further than to fit the case of a single bound state. More bound states can presumably be accounted for with each higher order of approximation.[10]

The Coulomb limit $\mu\rightarrow 0$ in (6.1) would be an interesting case, but, unfortunately, it cannot be handled owing to the "infrared" divergences which

[7] T. Regge, Nuovo Cimento **14**, 951 (1959).

[8] T. D. Lee, Phys. Rev. **45**, 1329 (1954).

[9] See, for example, Ref. 3, Chap. IV.

[10] This feature (that the higher the number of levels, the higher the order of approximation required to account for them) seems a basic rule of thumb in the present type of approach. This is another area where precise mathematical results would be desirable.

appear in the second Born approximation. We therefore restrict ourselves to finite-range forces. For simplicity we then use units such that $\mu=1$ and $m=\frac{1}{2}$.

A. Momentum Representation, Eq. (5.7)

In this representation, the constant-phase region consists of $-\infty<E<0$ for $\Delta^2\geqslant 0$. We want to investigate physical scattering ($E>0$). Hence, the best results should be obtained at small E, i.e., close to the constant-phase region. Application of (2.6) yields for the first two Born terms M_1 and M_2 of the scattering amplitude the expressions given in the Appendix. We shall be interested in the total cross section σ. (The differential cross section becomes isotropic near threshold.) Using the optical theorem,

$$\sigma=E^{-1/2}\,\mathrm{Im}M(0,E). \tag{6.3}$$

We have

$$\begin{aligned} &M_1(0,E)=-1,\\ &M_2(0,E)=(8\pi)^{-1}(1-2i\sqrt{E})^{-1}(E\geqslant 0). \end{aligned} \tag{6.4}$$

From (6.3) and (5.7),

$$\sigma=E^{-1/2}\,\mathrm{Im}\frac{-1}{\kappa_M+(8\pi)^{-1}(1-2i\sqrt{E})^{-1}} \tag{6.5}$$

or

$$\sigma=(4\pi\kappa_M{}^2)^{-1}[4E+(1+1/8\pi\kappa_M)^2]^{-1}. \tag{6.6}$$

In particular, at threshold,

$$\sigma_0=(4\pi\kappa_M{}^2)^{-1}(1+1/8\pi\kappa_M)^{-2}. \tag{6.7}$$

Eliminating κ_M in favor of the more physical σ_0, we obtain

$$\sigma=\sigma_0\{1+[1-2(4\pi/\sigma_0)^{1/2}(\sigma_0/4\pi)E\}^{-1}. \tag{6.8}$$

If σ_0 is large, this should be compared with the effective-range formula[11]

$$\sigma=\sigma_0\{1+[1-2.1196(4\pi/\sigma_0)^{1/2}](\sigma_0/4\pi)E\}^{-1}, \tag{6.9}$$

where the denominator is written to first order in E; the coefficient of $\sigma_0 E$ is written to first order in $\sigma_0{}^{-1/2}$. Comparison of (6.8) and (6.9) shows that the zero-range limits agree exactly, while the first-order range correction is about twice as large in our method as in the effective-range method.

We next show that our approximation is unitary at threshold, i.e., that an integration over angles of the differential cross section also yields the result (6.6) at zero energy. Since the amplitude is isotropic we use

$$d\sigma/d\Omega=(4\pi)^{-2}|M(0,0)|^2 \tag{6.10}$$

[a factor of $(\frac{1}{2})^2$ is due to $m=\frac{1}{2}$], where

$$M(0,0)=-(\kappa_M+1/8\pi)^{-1}; \tag{6.11}$$

Eq. (6.6) then follows.

[11] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Chap. II.

B. Partial-Wave Representation, Eq. (5.8)

Here the constant-phase region is $-\frac{1}{4}<E<0$. Equation (5.8) will be used to determine a single S wave bound state occurring in that region as a function of σ_0, another physical quantity connected with the positive end of the region. We first note that the representations (5.7) and (5.8) for $l=0$ coincide at threshold. We may use this as a consistency criterion to put

$$\kappa_N=\kappa_M. \tag{6.12}$$

In other words, σ_0 is now the same whether obtained from (5.7) (with or without the help of the optical theorem) or from (5.8). Setting $l=0$, we find the binding energy as the value of $-E$ for which the denominator of (5.8) vanishes. Thus, we must solve the equations

$$\kappa_M=N_2(0,E)/N_1(0,E) \tag{6.13}$$

and (6.7) for $E=-B$ as a function of σ_0. [The functions $N_{1,2}(0,E)$ are given in the Appendix.] The results are plotted in Fig. 1 and compared with the essentially exact results.[12] The present method does not begin to break down until $B\approx 0.20$, i.e., fairly close to the "far end", $B=0.25$, of the constant-phase region. In Fig. 1, the results are also compared with those of the effective-range analysis,[11] both in the zero-range limit and with first-order range connection. Near $B=0$ the latter approximation is

$$4\pi/\sigma_0=B-2.1196B^{3/2}+\cdots, \tag{6.14}$$

while the present method gives

$$4\pi/\sigma_0=B-(13/6)B^{3/2}+\cdots, \tag{6.15}$$

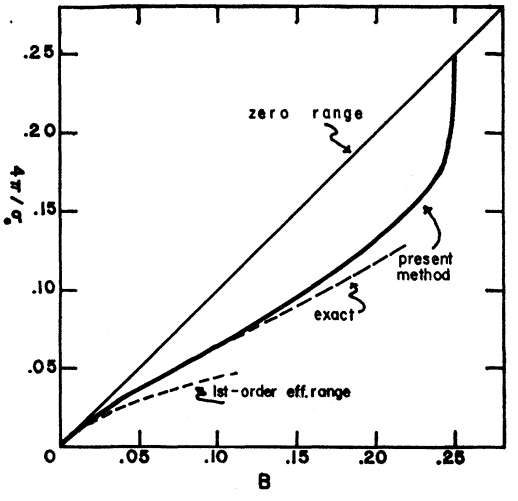

FIG. 1. Relation between the zero-energy cross section σ_0 and the binding energy B in the case of a Yukawa potential with a single shallow bound state, according to various approximation schemes. Concerning the "exact" curve, see Ref. 11.

[12] These represent a combination of the results of J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949) on σ_0 as a function of g, and of L. Hulthén and K. V. Laurikainen, Rev. Mod. Phys. **23**, 1 (1951) on B as a function of g.

a striking agreement to this order. In contrast, the agreement between (6.8) and (6.9) was not so good This may be due to the use of the optical theorem as a short cut in obtaining (6.8). Integration of $d\sigma/d\Omega$ should be considerably better, as experience with other methods indicates.

7. SUMMARY

A method has been presented for making "best" use of the information contained in the first few terms of the Born series. The region of applicability is, loosely speaking, the low-energy or low-binding-energy range. If the second Born approximation is given, then the method merely consists of a completely elementary algebraic rearrangement of the two Born terms, and is, therefore, equally applicable to field theory and to potential theory. The formula, which is of the denominator type made familiar through the Lee model and the renormalization-group treatment of the photon propagator, may give rise to bound states or resonances. It is very successful for the Yukawa potential in the one-level case. If higher Born terms are to be used, the method involves the solution of a nonlinear differential equation in one variable.

ACKNOWLEDGMENTS

I am much indebted to the advice and criticism of many theorists at Indiana University, particularly to Professor M. H. Ross.

APPENDIX: SECOND BORN APPROXIMATION FOR YUKAWA POTENTIAL SCATTERING

For completeness, we include the relevant formulas concerning the first Born terms in the case of a Yukawa potential [Eqs. (6.1) and (6.2)], even though some of them are well known. In the notation of Eqs. (5.4), (5.5), and (5.10), we have

$$M_1=-(\Delta^2+1)^{-1}, \tag{A1}$$

$$M_2=-(2\pi)^{-3}\int d^3k\{[(p+k)^2+1]\times[(q+k)^2+1][E-k^2+i\epsilon]\}^{-1}. \tag{A2}$$

The latter integral is conveniently evaluated by Feynman's method[13] to yield

$$M_2=(16\pi)^{-1}\int_0^1 xdx[\tfrac{1}{4}\Delta^2x^2-E(1-x)^2+x]^{-1}\times[-E(1-x)^2+x]^{-1/2}\,(E<0), \tag{A3}$$

analytic continuations being through the upper-half E plane in this and all subsequent formulas. This integral may be evaluated in terms of elementary functions,[1] a result not used in this paper. The special case $\Delta^2=0$ is simple to evaluate and yields (6.4).

[13] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Cambridge, Massachusetts, 1955), Appendix A5.

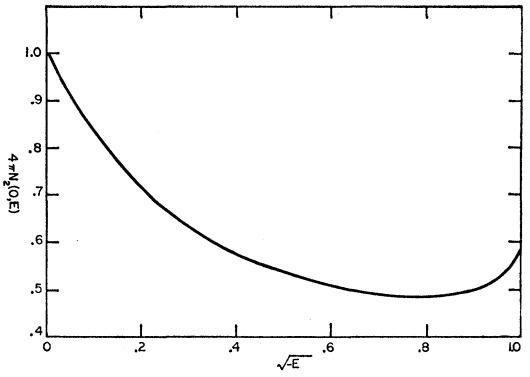

FIG. 2. The function $4\pi N_2(0,E)$, given by (A7), plotted against $(-E)^{1/2}$.

The S-wave amplitudes are given by

$$N_n(0,E)=\int_{-1}^{1} dz\, M_n(2E(1-z),E). \tag{A4}$$

We find

$$N_1(0,E)=-(2E)^{-1}\ln(4E+1)\ (E>-\tfrac{1}{4}), \tag{A5}$$

$$N_2(0,E)=(8\pi E)^{-1}\int_0^1 x^{-1}dx[-E(1-x)^2+x]^{-1/2}\times\ln\frac{x+E(2x-1)}{-E(1-x)^2+x}\ (-1<E<0). \tag{A6}$$

This integral may be evaluated in terms of the Spence function[14] L. The result is

$$N_2(0,E)=(16\pi)^{-1}(-E)^{-3/2}\times[L(X^2)+L(Y^2)-2L(XY)]\qquad(-1<E<0), \tag{A7}$$

where

$$X=[1+2(-E)^{1/2}]^{-1},\qquad Y=1-2(-E)^{1/2}, \tag{A8}$$

$$L(z)=-\int_0^z \zeta^{-1}d\zeta\,\ln|1-\zeta|\,. \tag{A9}$$

This result is useful in numerical work and is believed new. Near $E=0^-$, $N_2(0,E)$ may be expanded as a Taylor series in $(-E)^{1/2}$. The leading terms are

$$4\pi N_2(0,E)=1-2(-E)^{1/2}+(25/6)(-E)+\cdots. \tag{A10}$$

A plot of $4\pi N_2(0,E)$ is shown in Fig. 2.

Note added in proof. I am grateful to Dr. R. J. Eden for pointing out to me the interesting similarity of the second-order result (5.2) to a so-called Padé approximant.[15] It seems unlikely, however, that this similarity will persist to higher orders.

[14] For simple properties and a tabulation of $L(z)$, see K. Mitchell, Phil. Mag. **40**, 351 (1949).

[15] G. A. Baker, Jr., J. L. Gammel, and J. G. Wills, J. Math. Anal. and Appl. **2**, 405 (1961).